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10. Explicit higher local class field theory

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In this section we present an approach to higher local class field theory [F1-2] different from Kato's (see section 5) and Parshin's (see section 7) approaches.

Let F ($F = K_n, \dots, K_0$) be an n-dimensional local field. We use the results of section 6 and the notations of section 1.

10.1. Modified class formation axioms

Consider now an approach based on a generalization [F2] of Neukirch's approach [N].

Below is a modified system of axioms of class formations (when applied to topological K-groups) which imposes weaker restrictions than the classical axioms (cf. section 11).

(A1). *There is a $\hat{\mathbb{Z}}$-extension of F.*

In the case of higher local fields let F_{pur}/F be the extension which corresponds to K_0^{sep}/K_0: $F_{\text{pur}} = \cup_{(l,p)=1} F(\mu_l)$; the extension F_{pur} is called the *maximal purely unramified extension* of F. Denote by Frob_F the lifting of the Frobenius automorphisms of K_0^{sep}/K_0. Then

$$\text{Gal}(F_{\text{pur}}/F) \simeq \hat{\mathbb{Z}}, \quad \text{Frob}_F \mapsto 1.$$

(A2). *For every finite separable extension F of the ground field there is an abelian group A_F such that $F \to A_F$ behaves well* (*is a Mackey functor, see for instance* [D]; *in fact we shall use just topological K-groups*) *and such that there is a homomorphism $\mathfrak{v}\colon A_F \to \mathbb{Z}$ associated to the choice of the $\hat{\mathbb{Z}}$-extension in* (A1) *which satisfies*

$$\mathfrak{v}(N_{L/F} A_L) = |L \cap F_{\text{pur}} : F| \ \mathfrak{v}(A_F).$$

In the case of higher local fields we use the valuation homomorphism

$$\mathfrak{v}\colon K_n^{\text{top}}(F) \to \mathbb{Z}$$

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of 6.4.1. From now on we write $K_n^{\mathrm{top}}(F)$ instead of A_F. The kernel of $\mathfrak{v}$ is $VK_n^{\mathrm{top}}(F)$.

Put

$$\mathfrak{v}_L = \frac{1}{|L \cap F_{\mathrm{pur}} : F|}\mathfrak{v} \circ N_{L/F}.$$

Using (A1), (A2) for an arbitrary finite Galois extension L/F define the *reciprocity map*

$$\Upsilon_{L/F}\colon \mathrm{Gal}(L/F) \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L), \qquad \sigma \mapsto N_{\Sigma/F}\Pi_\Sigma \mod N_{L/F}K_n^{\mathrm{top}}(L)$$

where Σ is the fixed field of $\widetilde{\sigma}$ and $\widetilde{\sigma}$ is an element of $\mathrm{Gal}(L_{\mathrm{pur}}/F)$ such that $\widetilde{\sigma}|_L = \sigma$ and $\widetilde{\sigma}|_{F_{\mathrm{pur}}} = \mathrm{Frob}_F^i$ with a positive integer i. The element Π_Σ of $K_n^{\mathrm{top}}(\Sigma)$ is any such that $\mathfrak{v}_\Sigma(\Pi_\Sigma) = 1$; it is called a *prime element* of $K_n^{\mathrm{top}}(\Sigma)$. This map doesn't depend on the choice of a prime element of $K_n^{\mathrm{top}}(\Sigma)$, since $\Sigma L/\Sigma$ is purely unramified and $VK_n^{\mathrm{top}}(\Sigma) \subset N_{\Sigma L/\Sigma}VK_n^{\mathrm{top}}(\Sigma L)$.

(A3). *For every finite subextension L/F of F_{pur}/F (which is cyclic, so its Galois group is generated by, say, a σ)*

- (A3a) $|K_n^{\mathrm{top}}(F) : N_{L/F}K_n^{\mathrm{top}}(L)| = |L : F|$;
- (A3b) $0 \longrightarrow K_n^{\mathrm{top}}(F) \xrightarrow{i_{F/L}} K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L)$ *is exact*;
- (A3c) $K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$ *is exact.*

Using (A1), (A2), (A3) one proves that $\Upsilon_{L/F}$ is a homomorphism [F2].

(A4). *For every cyclic extensions L/F of prime degree with a generator σ and a cyclic extension L'/F of the same degree*

- (A4a) $K_n^{\mathrm{top}}(L) \xrightarrow{1-\sigma} K_n^{\mathrm{top}}(L) \xrightarrow{N_{L/F}} K_n^{\mathrm{top}}(F)$ *is exact*;
- (A4b) $|K_n^{\mathrm{top}}(F) : N_{L/F}K_n^{\mathrm{top}}(L)| = |L : F|$;
- (A4c) $N_{L'/F}K_n^{\mathrm{top}}(L') = N_{L/F}K_n^{\mathrm{top}}(L) \Rightarrow L = L'$.

If all axioms (A1)–(A4) hold then the homomorphism $\Upsilon_{L/F}$ induces an isomorphism [F2]

$$\Upsilon_{L/F}^{\mathrm{ab}}\colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L).$$

The method of the proof is to define explicitly (as a generalization of Hazewinkel's approach [H]) a homomorphism

$$\Psi_{L/F}^{\mathrm{ab}}\colon K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

and then show that $\Psi_{L/F}^{\mathrm{ab}} \circ \Upsilon_{L/F}^{\mathrm{ab}}$ is the indentity.

10.2. Characteristic p case

Theorem 1 ([F1], [F2]). *In characteristic p all axioms (A1)–(A4) hold. So we get the reciprocity map $\Psi_{L/F}$ and passing to the limit the reciprocity map*

$$\Psi_F\colon K_n^{\text{top}}(F) \to \text{Gal}(F^{\text{ab}}/F).$$

Proof. See subsection 6.8. (A4c) can be checked by a direct computation using the proposition of 6.8.1 [F2, p. 1118–1119]; (A3b) for p-extensions see in 7.5, to check it for extensions of degree prime to p is relatively easy [F2, Th. 3.3]. $\square$

Remark. Note that in characteristic p the sequence of (A3b) is not exact for an arbitrary cyclic extension L/F (if $L \not\subset F_{\text{pur}}$). The characteristic zero case is discussed below.

10.3. Characteristic zero case. I

10.3.1. prime-to-p-part.

It is relatively easy to check that all the axioms of 10.1 hold for prime-to-p extensions and for

$$K_n'(F) = K_n^{\text{top}}(F)/VK_n^{\text{top}}(F)$$

(note that $VK_n^{\text{top}}(F) = \bigcap_{(l,p)=1} lK_n^{\text{top}}(F)$). This supplies the prime-to-p-part of the reciprocity map.

10.3.2. p-part.

If $\mu_p \leqslant F^*$ then all the axioms of 10.1 hold; if $\mu_p \nleqslant F^*$ then everything with exception of the axiom (A3b) holds.

Example. Let $k = \mathbb{Q}_p(\zeta_p)$. Let $\omega \in k$ be a p-primary element of k which means that $k(\sqrt[p]{\omega})/k$ is unramified of degree p. Then due to the description of K_2 of a local field (see subsection 6.1 and [FV, Ch.IX §4]) there is a prime elements π of k such that $\{\omega, \pi\}$ is a generator of $K_2(k)/p$. Since $\alpha = i_{k/k(\sqrt[p]{\omega})}\{\omega,\pi\} \in pK_2(k(\sqrt[p]{\omega}))$, the element α lies in $\bigcap_{l\geqslant 1} lK_2(k(\sqrt[p]{\omega}))$. Let $F = k\{\{t\}\}$. Then $\{\omega,\pi\} \notin pK_2^{\text{top}}(F)$ and $i_{F/F(\sqrt[p]{\omega})}\{\omega,\pi\} = 0$ in $K_2^{\text{top}}(F(\sqrt[p]{\omega}))$.

Since all other axioms are satisfied, according to 10.1 we get the reciprocity map

$$\Upsilon_{L/F}\colon \text{Gal}(L/F) \to K_n^{\text{top}}(F)/N_{L/F}K_n^{\text{top}}(L), \quad \sigma \mapsto N_{\Sigma/F}\Pi_\Sigma$$

for every finite Galois p-extension L/F.

To study its properties we need to introduce the notion of Artin–Schreier trees (cf. [F3]) as those extensions in characteristic zero which in a certain sense come from characteristic p. Not quite precisely, there are sufficiently many finite Galois p-extensions for which one can directly define an explicit homomorphism

$$K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$$

and show that the composition of $\Upsilon_{L/F}^{\mathrm{ab}}$ with it is the identity map.

10.4. Characteristic zero case. II: Artin–Schreier trees

10.4.1.

Definition. A p-extension L/F is called an *Artin–Schreier tree* if there is a tower of subfields $F = F_0 - F_1 - \cdots - F_r = L$ such that each F_i/F_{i-1} is cyclic of degree p, $F_i = F_{i-1}(\alpha)$, $\alpha^p - \alpha \in F_{i-1}$.

A p-extension L/F is called a *strong Artin–Schreier tree* if every cyclic subextension M/E of degree p, $F \subset E \subset M \subset L$, is of type $E = M(\alpha)$, $\alpha^p - \alpha \in M$.

Call an extension L/F *totally ramified* if $f(L|F) = 1$ (i.e. $L \cap F_{\mathrm{pur}} = F$).

Properties of Artin–Schreier trees.

(1) if $\mu_p \not\leqslant F^*$ then every p-extension is an Artin–Schreier tree; if $\mu_p \leqslant F^*$ then $F(\sqrt[p]{a})/F$ is an Artin–Schreier tree if and only if $aF^{*p} \leqslant V_F F^{*p}$.

(2) for every cyclic totally ramified extension L/F of degree p there is a Galois totally ramified p-extension E/F such that E/F is an Artin–Schreier tree and $E \supset L$.

For example, if $\mu_p \leqslant F^*$, F is two-dimensional and t_1, t_2 is a system of local parameters of F, then $F(\sqrt[p]{t_1})/F$ is not an Artin–Schreier tree. Find an $\varepsilon \in V_F \setminus V_F^p$ such that M/F ramifies along t_1 where $M = F(\sqrt[p]{\varepsilon})$. Let $t_{1,M}, t_2 \in F$ be a system of local parameters of M. Then $t_1 t_{1,M}^{-p}$ is a unit of M. Put $E = M\big(\sqrt[p]{t_1 t_{1,M}^{-p}}\big)$. Then $E \supset F(\sqrt[p]{t_1})$ and E/F is an Artin–Schreier tree.

(3) Let L/F be a totally ramified finite Galois p-extension. Then there is a totally ramified finite p-extension Q/F such that LQ/Q is a strong Artin–Schreier tree and $L_{\mathrm{pur}} \cap Q_{\mathrm{pur}} = F_{\mathrm{pur}}$.

(4) For every totally ramified Galois extension L/F of degree p which is an Artin–Schreier tree we have

$$\mathfrak{v}_{L_{\mathrm{pur}}}(K_n^{\mathrm{top}}(L_{\mathrm{pur}})^{\mathrm{Gal}(L/F)}) = p\mathbb{Z}$$

where $\mathfrak{v}$ is the valuation map defined in 10.1, $K_n^{\mathrm{top}}(L_{\mathrm{pur}}) = \varinjlim_M K_n^{\mathrm{top}}(M)$ where M/L runs over finite subextensions in L_{pur}/L and the limit is taken with respect to the maps $i_{M/M'}$ induced by field embeddings.

Proposition 1. *For a strong Artin–Schreier tree* L/F *the sequence*

$$1 \to \mathrm{Gal}(L/F)^{\mathrm{ab}} \xrightarrow{g} VK_n^{\mathrm{top}}(L_{\mathrm{pur}})/I(L|F) \xrightarrow{N_{L_{\mathrm{pur}}/F_{\mathrm{pur}}}} VK_n^{\mathrm{top}}(F_{\mathrm{pur}}) \to 0$$

is exact, where $g(\sigma) = \sigma\Pi - \Pi,\ \mathfrak{v}_L(\Pi) = 1,\ I(L|F) = \langle \sigma\alpha - \alpha : \alpha \in VK_n^{\mathrm{top}}(L_{\mathrm{pur}})\rangle$*.*

Proof. Induction on $|L : F|$ using the property $N_{L_{\mathrm{pur}}/M_{\mathrm{pur}}} I(L|F) = I(M|F)$ for a subextension M/F of L/F. □

10.4.2. As a generalization of Hazewinkel's approach [H] we have

Corollary. *For a strong Artin–Schreier tree* L/F *define a homomorphism*

$$\Psi_{L/F} \colon VK_n^{\mathrm{top}}(F)/N_{L/F}VK_n^{\mathrm{top}}(L) \to \mathrm{Gal}(L/F)^{\mathrm{ab}}, \quad \alpha \mapsto g^{-1}((\mathrm{Frob}_L - 1)\beta)$$

where $N_{L_{\mathrm{pur}}/F_{\mathrm{pur}}}\beta = i_{F/F_{\mathrm{pur}}}\alpha$ *and* Frob_L *is defined in 10.1.*

Proposition 2. $\Psi_{L/F} \circ \Upsilon^{\mathrm{ab}}_{L/F} \colon \mathrm{Gal}(L/F)^{\mathrm{ab}} \to \mathrm{Gal}(L/F)^{\mathrm{ab}}$ *is the identity map; so for a strong Artin–Schreier tree* $\Upsilon^{\mathrm{ab}}_{L/F}$ *is injective and* $\Psi_{L/F}$ *is surjective.*

Remark. As the example above shows, one cannot define $\Psi_{L/F}$ for non-strong Artin–Schreier trees.

Theorem 2. $\Upsilon^{\mathrm{ab}}_{L/F}$ *is an isomorphism.*

Proof. Use property (3) of Artin–Schreier trees to deduce from the commutative diagram

$$\begin{array}{ccc}
\mathrm{Gal}(LQ/Q) & \xrightarrow{\Upsilon_{LQ/Q}} & K_n^{\mathrm{top}}(Q)/N_{LQ/Q}K_n^{\mathrm{top}}(LQ) \\
\downarrow & & \downarrow{\scriptstyle N_{Q/F}} \\
\mathrm{Gal}(L/F) & \xrightarrow{\Upsilon_{L/F}} & K_n^{\mathrm{top}}(F)/N_{L/F}K_n^{\mathrm{top}}(L)
\end{array}$$

that $\Upsilon_{L/F}$ is a homomorphism and injective. Surjectivity follows by induction on degree. □

Passing to the projective limit we get the reciprocity map

$$\Psi_F \colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

whose image in dense in $\mathrm{Gal}(F^{\mathrm{ab}}/F)$.

Remark. For another slightly different approach to deduce the properties of $\Upsilon_{L/F}$ see [F1].

10.5

Theorem 3. *The following diagram is commutative*

$$\begin{array}{ccc} K_n^{\mathrm{top}}(F) & \xrightarrow{\Psi_F} & \mathrm{Gal}(F^{\mathrm{ab}}/F) \\ {\scriptstyle\partial}\big\downarrow & & \big\downarrow \\ K_{n-1}^{\mathrm{top}}(K_{n-1}) & \xrightarrow{\Psi_{K_{n-1}}} & \mathrm{Gal}(K_{n-1}^{\mathrm{ab}}/K_{n-1}). \end{array}$$

Proof. Follows from the explicit definition of $\Upsilon_{L/F}$, since $\partial\{t_1, \dots, t_n\}$ is a prime element of $K_{n-1}^{\mathrm{top}}(K_{n-1})$. □

Existence Theorem ([F1-2]). *Every open subgroup of finite index in $K_n^{\mathrm{top}}(F)$ is the norm group of a uniquely determined abelian extension L/F.*

Proof. Let N be an open subgroup of $K_n^{\mathrm{top}}(F)$ of prime index l.

If $p \neq l$, then there is an $\alpha \in F^*$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to $t^{(q-1)/l}$ where t is the tame symbol defined in 6.4.2.

If $\mathrm{char}(F) = p = l$, then there is an $\alpha \in F$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to $(\ ,\ ]_1$ defined in 6.4.3.

If $\mathrm{char}(F) = 0, l = p$, $\mu_p \leqslant F^*$, then there is an $\alpha \in F^*$ such that N is the orthogonal complement of $\langle\alpha\rangle$ with respect to V_1 defined in 6.4.4 (see the theorems in 8.3). If $\mu_p \nleqslant F^*$ then pass to $F(\mu_p)$ and then back to F using $(|F(\mu_p) : F|, p) = 1$.

Due to Kummer and Artin–Schreier theory, Theorem 2 and Remark of 8.3 we deduce that $N = N_{L/F} K_n^{\mathrm{top}}(L)$ for an appropriate cyclic extension L/F.

The theorem follows by induction on index. □

Remark 1. From the definition of K_n^{top} it immediately follows that open subgroups of finite index in $K_n(F)$ are in one-to-one correspondence with open subgroups in $K_n^{\mathrm{top}}(F)$. Hence the correspondence $L \mapsto N_{L/F} K_n(L)$ is a one-to-one correspondence between finite abelian extensions of F and open subgroups of finite index in $K_n(F)$.

Remark 2. If K_0 is perfect and not separably p-closed, then there is a generalization of the previous class field theory for totally ramified p-extensions of F (see Remark in 16.1). There is also a generalization of the existence theorem [F3].

Corollary 1. *The reciprocity map $\Psi_F\colon K_n^{\mathrm{top}}(F) \to \mathrm{Gal}(L/F)$ is injective.*

Proof. Use the corollary of Theorem 1 in 6.6. □

Corollary 2. *For an element* $\Pi \in K_n^{\text{top}}(F)$ *such that* $\mathfrak{v}_F(\Pi) = 1$ *there is an infinite abelian extension* F_Π/F *such that*

$$F^{\text{ab}} = F_{\text{pur}} F_\Pi, \quad F_{\text{pur}} \cap F_\Pi = F$$

and $\Pi \in N_{L/F} K_n^{\text{top}}(L)$ *for every finite extension* L/F, $L \subset F_\Pi$.

Problem. Construct (for $n > 1$) the extension F_Π explicitly?

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